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QUANTUM GRAVITY on the
CLASSICAL BACKGROUND: GROUP ANALYSIS, Part I

1 Introduction

In our article we present an approach to the description of gravitational field from the point of view of quantum field theory. We consider quantization in the neighbourhood of solution of Einstein equation and use the method of Bogoliubov transformations for this purpose.

Quantum gravity is a pure theory and couldn't be tested by laboratory experiments and astrophysical data, because in observable processes of Universe quantum effects connected with gravity are extremely small. In the same time gravitational interaction is universal one for all kinds of matter independently from its properties. Quantum gravity is not constructed yet, and in the present moment there are some relatively independent directions:

- quantum theory of gravitational field;
- theory of nongravitational field in the curved space-time;
- quantum cosmology and quantum theory of black holes;
- quantum supergravity and multidimensional theory of unification of interactions.

Quantum theory of gravitational field is based on the quantization of classical theory of gravitational field - general relativity.

Theory of the field in the curved space-time investigates the methods of quantization of matter fields on the background of classical gravitational field.

Quantum cosmology is application of methods of quantum gravity to the Early Universe in the vicinity of singularity. The most important achievement of quantum cosmology is construction of concrete models of inflationary universe.

Quantum theory of Black holes investigates mainly the effects of particles creations and vacuum polarization of the gravitational field of black holes.

Quantum gravity is close to the multidimensional theories of Grand Unification. Unification of space-time symmetries with inner symmetries and gauge symmetries of strong and electroweak interactions is reached due to introduction of curved space-time of $4+d$ dimensions. The symmetry of this space-time determines the symmetry of interactions.

In future all of those directions are believed to be the parts of Unified quantum gravity theory. That is why the appropriate choice of the methods of quantization of gravitational field seems to be very important.

Quantization on the classical background seems to be a very special case, but existence of classical component is an essential feature of gravitational field and our choice of quantization looks to be reasonable.

We consider exact solutions of Einstein equation to be classical background and quantize in the neighbourhood of this exact solutions.

Any space-time could be treated as a solution of Einstein equation

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda_{ab} = 8\pi T_{ab}$$

with the appropriate choice of energy momentum tensor T_{ab} of some concrete form of the matter.

However it is possible to find exact solutions only for the space with a rather high degree of symmetry. Apart from this fact exact solutions describes some kind of ideal situation: any origin of space-time contains different types of matter. And it is possible to obtain exact solutions only in the case of simple enclosed matter.

Nevertheless exact solutions are important, because they give ideas concerning qualificationly new phenomena that could arise in the general relativity and hence point on the possible properties of real solutions of field equations. The following examples shows a lot of interesting types of solutions.

a) The simplest metrix has constant curvature. It means that the space is homogeneous.

If $R > 0$ we have De Sitter space-time of the first type, that describes space-time of stationary Universe model.

If we believe that our Universe is approximately homogeneous and spherically symmetrical in large scale, than we have a model of Robertson-Walker (or Freedman) and the metrix reads:

$$ds^2 = -dt^2 + S^2(t)d\sigma^2,$$

here $d\sigma^2$ is 3D metrix of space with constant curvature, the geometry of space depends on sine of 3D curvature.

Robertson-Walker solution symmetry demands energy -momentum tensor to be the tensor of ideal fluid that describe matter in Universe spread homogeneously.

Large-scale solutions are good model for large scale matter distribution, but for the single objects like stars we have to use asymptotically flat solutions.

b) This geometry with good approximations could be described by well known Schwartzshild solution that represent spherically symmetrical out space of massive body.

$$ds^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

c) Out space-time of charged spherically symmetrical body is described by Reisner-Nordstrom solution. (Though it has no spin neither magnetic angular momentum hence it couldn't be good description of gravitational field of electron.) This is unique asymptotically flat solution of Einstein-Maxwell equation:

$$ds^2 = - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

d) Kerr solution describe stationary axial-symmetrical asymptotically flat field outside the rotating massive object:

$$ds^2 = \rho^2 \left(\frac{dr^2}{\Delta} + d\Theta^2 \right) + (r^2 + a^2) \sin^2 \theta d\Phi - dt^2 + \frac{2mr}{\rho^2} (a \sin^2 \Theta d\Phi - dt)^2$$

Kerr solution seems to be unequally possible solution for black holes.

e) Another interesting solution is a solution of flat waves in the empty space-time and the metrix reads:

$$ds^2 = 2dudv + dy^2 + dz^2 + H(x, y, z) du^2$$

Those physically interesting exact solutions could be chosen as a classical background. There are a lot of other exact solutions but not much of them are studied perfectly.

Now we would like to give some arguments why the method of Bogoliubov group variables have advantages while we quantize the field with some type of continuous symmetry on the classical background: application of Bogoliubov

group variables permits to take into account conservation laws and avoid zero-mode problem.

Conservation laws are the fundamental principles that connected directly with symmetry properties of the system under consideration. Bogoliubov method permits to take into account conservation laws precisely and explicitly in the any order of perturbations theory construction.

The main idea of the method consists in using new variables that has sense of generators of system symmetry group. New variables are cyclic while their canonical momenta turns out to be integrals of motion and hence commute with Hamiltonian providing exact performance of conservation laws. In the case of not weak interactions the perturbation theory as any approximation method could violate exact performance of conservation laws and we can't evaluate how our approximation of state vector is close to the real state vector. Taking into account this circumstance we can state that Bogoliubov method has significant advantage.

The second dignity of this method application is the following: we can avoid zero-mode problem that appears in the process of quantization of the system that has continuous symmetry.

In our resent articles we have developed the Bogoliubov method for any relativistic system that allows existence of symplectic structures, one of such systems is gravitational field.

Now we would like to represent the scheme of quantization of gravitational field on the classical background by means of Bogoliubov group variables.

2 Space-time description

We consider gravitational field in (3+1)-dimensioned formalism that has been proposed by Arnowitt-Deser-Misner (ADM).

Metrical tensor in this formalism looks like

$$g_{\alpha\beta} = \begin{pmatrix} -a^2 + b^t b_t & b^t \\ b_t & \gamma_{st} \end{pmatrix}$$

here γ_{st} is metrix of 3D-space in 4D-manyfold.

Canonical momentum π^{st} is determined as usual:

$$\pi^{st} = -\sqrt{\gamma} \left(K^{st} - \gamma^{st} K \right),$$

here

$$\sqrt{\gamma} = \sqrt{\det \|\gamma_{st}\|}; \quad K_{tp} = -a\Gamma_{tp}^0;$$

$$\Gamma_{tp}^s = \frac{1}{2}\gamma^{sp}\Gamma_{tlp}; \quad \Gamma_{tlp} = \gamma_{pl,t} + \gamma_{pt,l} - \gamma_{tl,p}.$$

Denoting as usually

$$R_{\kappa\lambda} = \Gamma_{\kappa\lambda,\sigma}^\sigma - \Gamma_{\kappa\sigma,\lambda}^\sigma + \Gamma_{\kappa\lambda}^\sigma \Gamma_{\sigma\rho}^\rho - \Gamma_{\kappa\rho}^\sigma \Gamma_{\lambda\sigma}^\rho,$$

we can represent the action of gravitational field

$$S = \int d^3x \sqrt{g} g^{\kappa\lambda} R_{\kappa\lambda}$$

in the following form:

$$S = \int d^3x \left(\pi^{st} \gamma_{st,0} - aH - b_s H^s \right),$$

here

$$H = \frac{1}{\sqrt{\gamma}} \left(\pi_{st} \pi^{st} - \frac{1}{2} \pi^2 \right) - \sqrt{\gamma} R,$$

$$H^s = -2\pi_{,l}^{sl}.$$

Suppose that 4D manifold with given metrix permits to chose space-like hypersurface Σ and to set normals field on this hypersurface. Those normals are tangent to geodesic and determine time coordinate. Hence geometry of 4D manifold could be described via Gaussian coordinates:

$$g_{\alpha\beta} = \begin{pmatrix} -a^2 & 0 \\ 0 & \gamma_{st} \end{pmatrix}$$

We have to chose hypersurface Σ in according with the folloowing principle:

Suppose that normal \vec{n} is given in the hypersurface Σ at the point X . Let's chose arbitrary closed line $K \subset \Sigma$. ($X \in K$)

Normal vector \vec{n} have to coincide with it's original direction after whole cycle motion along the line K - this is criterium of the hypersurface Σ choice. Choquet-Bruhat showed that this criterium could be realized via imposing of the Hamiltonian constraint:

$$\frac{1}{\sqrt{\gamma}} \left(\pi_{st} \pi^{st} - \frac{1}{2} \pi^2 \right) \sqrt{\gamma} R = 0$$

and momentum constraint:

$$\pi_{,l}^{sl} = 0.$$

General principles of canonical formalism for the systems with constraints leads us to the following statement (Lichnerovitch, Choquet-Bruhat, Dirac, Antrowitt-Deser-Misner):

If the evolutions equations

$$\begin{aligned} \gamma_{st,0} &= \frac{2a}{\sqrt{\gamma}} \left(\pi_{st} - \frac{1}{2} \gamma_{st} \pi \right) + b_{s;t} + b_{t;s}; \\ \pi_{,o}^{st} &= -a \sqrt{\gamma} \left(R^{st} - \frac{1}{2} \gamma^{st} R \right) + \frac{a}{2\sqrt{\gamma}} \left(\pi_{st} \pi^{st} - \frac{1}{2} \pi^2 \right) \gamma^{st} + \\ &\quad - \frac{a}{2\sqrt{\gamma}} \left(\pi_l^s \pi^{lt} - \frac{1}{2} \pi^{st} \pi \right) + \sqrt{\gamma} \left(\gamma^{sl} c_{;l}^t - \gamma^{st} c_{;l}^l \right) + \\ &\quad + \left(\pi^{st} b^l \right)_{;l} - \pi^{sl} b_{;l}^t - \pi^{lt} b_{;l}^s; \quad c^l = \gamma^{ls} a_{;s} \end{aligned}$$

are performed on the 3D-space, then in 4D manifold the Einstein equations holds true:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \quad \left(R_{ikl;m}^n + R_{imk;l}^n + R_{ilm;k}^n = 0 \right).$$

3 Bogoliubov Group Variables

Let's variables x' are connected with x by the following group transformation:

$$x' = f(x, a), \quad x'' = f(f(x, a), b) = f(x, c), \quad c = \varphi(a, b).$$

Under variation of group parameters a variation of coordinates reads:

$$(\delta x')^i = \xi_s^i(x') B_p^s(a) \delta a^p,$$

here

$i = 0, 1, 2, 3$ - the number of coordinates,

$p = 1, \dots, r$, where r is a quantity of group generators,

$B_p^s(a)$ defines group properties.

Note that conservation laws performance in curved space-time is connected with Killing vectors existence, they are not straightforward sequence of system space-time transformation invariance.

In present case Bogoliubov transformation reconstruct invariance with respect to transformation group, that has been violated due to explicit extraction of classical field.

It means the following: if we have made quantization in some surface Σ in definite moment, application of group variables permits to state that we can move this surface Σ with according to group laws (including transformation that changes the time coordinate).

Let's consider couples of functions $f_{st}(x)$, $f_n^{st}(x)$ and define Bogoliubov transformation as following:

$$f_{st}(x) = g v_{st}(x') + u_{st}(x'), \quad f_n^{st}(x) = g v_n^{st}(x') + u_n^{st}(x'), \quad (2)$$

dimensionless parameter g is assumed to be large, and group parameters a^p are independent new variables.

The substitution $f_{st}(x) \rightarrow \{u_{st}, (a)\}$ enlarge the number of independent variables on r , so problem is how to formulate invariant conditions, which we have to impose on functions $u_{st}(x')$ in order to equalize the number of independent variables in the both part of equation (2).

We consider systems in which there are invariant symplectic forms that looks like the following:

$$\omega(u_{st}, N^{stp}) = \int_{\Sigma} (u_n^{st}(x') N_{st}^p(x') - u_{st}(x') N_n^{stp}(x')) d\sigma,$$

here Σ is some space-like surface. Everywhere in the article we mean summation with respect to all s and t .

Additional conditions are:

$$\omega(u_{st}, N^{stp}) = 0. \quad (3)$$

We chose some functions $N_{st}^p(x')$ ($p = 1 \dots r$, the quantity functions is equal to the quantity of new independent variables)

It is possible to obtain equations, which define group variables as functional of $f_{st}(x)$ and $f_n^{st}(x)$ on the Σ , by substitution $u_{st}(x')$ in the additional conditions in according with (2).

Cause the forms $\omega(N_{st}^a, u_{st})$ are invariant with respect to variations of a^p one can obtain:

$$-\int_{\Sigma} d\sigma \left(\delta f_{st}(x) N_n^{stk}(x') - \delta f_n^{st}(x) N_{st}^k(x') \right) - g B_p^s(a) \delta a^p - B_p^r(a) \delta a^p R_r^s = 0, \quad (4)$$

here

$$R_r^k = \int_{\Sigma} d\sigma \xi_r^i(x') \left(N_{ni}^{stk}(x') u_{st}(x') - N_{sti}^k(x') u_n^{st}(x') \right).$$

It is useful to formulate equation (4) in the differential form:

$$\frac{\delta a^p}{\delta f_{st}(x)} = -\frac{1}{g} A_k^p(a) N_n^{stk}(x') - \frac{1}{g} \frac{\delta a^q}{\delta f_{st}(x)} B_q^r(a) R_r^s A_s^p(a),$$

$$\frac{\delta a^p}{\delta f_n^{st}(x)} = \frac{1}{g} A_k^p(a) N_{st}^k(x') - \frac{1}{g} \frac{\delta a^q}{\delta f_n^{st}(x)} B_q^r(a) R_r^s A_s^p(a),$$

$A_s^p(a)$ denote the matrix inverse to $B_q^s(a)$:

$$B_q^s(a) A_s^p(a) = \delta_q^p.$$

Denote

$$N_{st}^k(x') T_k^l = D_{st}^l(x')$$

$$N_n^{stk}(x') T_k^l = D_n^{stl}(x')$$

where T_s^l are the solution of the equation:

$$T_s^l = \delta_s^l - \frac{1}{g} T_s^r R_r^l.$$

So we can state that

$$\frac{\delta a^p}{\delta f_{st}(x)} = \frac{1}{g} A_l^p(a) D_{st}^{tl}(x'), \quad \frac{\delta a^p}{\delta f_n^{st}(x)} = -\frac{1}{g} A_l^p(a) D_{st}^l(x').$$

As a consequence of (3) we obtain linear dependence between derivatives with respect to $u_{st}(x')$ and $u_n^{st}(x')$:

$$\int_{\Sigma} d\sigma \left(M_{str}(x') \frac{\delta}{\delta u_{st}(x')} + M_{nr}^{st}(x') \frac{\delta}{\delta u_n^{st}(x')} \right) = 0,$$

where we define

$$\xi_r^i(x') v_{sti}(x') = M_{str}(x'), \quad \xi_r^i(x') v_{ni}^{st}(x') = M_{nr}^{st}(x').$$

(Here we demand the following relationship to be true:

$$\omega(M_{sta} N_{st}^k) = \delta_a^k)$$

Straightforward calculations give us $f_{st}(x)$ and $f_n^{st}(x)$ in the terms of new variables:

$$\frac{\delta}{\delta f_{st}(x)} = \frac{\delta}{\delta u_{st}(x')} + B_q^p(a) \frac{\delta a^q}{\delta f_{st}(x)} \left(S_p + A_p^r(a) \frac{\partial}{\partial a^r} \right),$$

$$\frac{\delta}{\delta f_n^{st}(x)} = \frac{\delta}{\delta u_n^{st}(x')} + B_q^p(a) \frac{\delta a^q}{\delta f_n^{st}(x)} \left(S_p + A_p^r(a) \frac{\partial}{\partial a^r} \right),$$

here S_p is defined as

$$-\int_{\Sigma} d\sigma \xi_p^i(x') \left(u_{sti}(x') \frac{\delta}{\delta u_{st}(x')} + u_{ni}^{st}(x') \frac{\delta}{\delta u_n^{st}(x')} \right) = S_p.$$

4 Secondary Quantization.

The operators $\hat{q}_{st}(x)$ and $\hat{p}^{st}(x')$:

$$\hat{q}_{st}(x) = \frac{1}{\sqrt{2}} \left(f_{st}(x) + i \frac{\delta}{\delta f_n^{st}(x)} \right), \quad \hat{p}^{st}(x) = \frac{1}{\sqrt{2}} \left(f_n^{st}(x) - i \frac{\delta}{\delta f_{st}(x)} \right), \quad (5)$$

are defined in the space L of functionals F where the scalar product defines as

$$\langle F_1 | F_2 \rangle = \int Df_{st} Df_n^{st} F_{1n}[f_{st}, f_n^{st}] F_2[f_{st}, f_n^{st}].$$

The operators (5) are self-conjugated in this space. They satisfy the formal commutation relation:

$$[\hat{q}_{st}(x), \hat{p}^{st}(x')] = i\delta(x - x').$$

So we can treat $\hat{q}_{st}(x)$ and $\hat{p}^{st}(x)$ as operators of coordinate and momentum of oscillators of field and we can develop the secondary quantization scheme. But straightforward use of this procedure leads us to the doubling of numbers of possible field states, because there are self-conjugate operators

$$\tilde{q}_{st}(x) = \frac{1}{\sqrt{2}} \left(f_{st}(x) - i \frac{\delta}{\delta f_n^{st}(x)} \right), \quad \tilde{p}^{st}(x) = \frac{1}{\sqrt{2}} \left(f_n^{st}(x) + i \frac{\delta}{\delta f_{st}(x)} \right),$$

that satisfy the same commutation relation and commute with $\hat{q}(x)$ and $\hat{p}(x)$.

The operators $\tilde{p}(x)$ and \tilde{q} also generate some operators of coordinate and momentum, which are defined in the Fock space orthogonal one described above. So quantization based on the (5) needs further reduction of states number. One of the reduction method is the holomorphic representation, for example.

We can consider the space of functionals $F[z_{st}, z^{st*}]$ isomorphic to the space $F[f_{st}, f_n^{st}]$. If we define

$$z_{st}(x) = f_{st}(x) + i f_n^{st}(x), \quad z^{st*}(x) = f_{st}(x) - i f_n^{st}(x),$$

so $\hat{q}_{st}(x)$ and $\hat{p}^{st}(x)$ can be defined as operators:

$$\hat{q}_{st}(x) = \frac{1}{\sqrt{2}} \left(z_{st}(x) - \frac{\delta}{\delta z^{st*}(x)} \right), \quad \hat{p}^{st}(x) = \frac{1}{\sqrt{2}} \left(z^{st*}(x) + \frac{\delta}{\delta z_{st}(x)} \right).$$

In the space $F[z_{st}, z^{st*}]$ reduction of states numbers is made by the choice state vector in the form:

$$F = \left(\exp \int_{\Sigma} z_{st}(x) z^{st*}(x) d\sigma \right) \Phi[z].$$

It is easy to see that vector

$$F_0 = \exp \left(\int_{\Sigma} z_{st}(x) z^{st*}(x) d\sigma \right)$$

is the vacuum of operators

$\hat{q}_{st}(x) = \frac{1}{\sqrt{2}} \left(z_{st}(x) - \frac{\delta}{\delta z^{st*}(x)} \right)$, $\hat{p}^{st}(x) = \frac{1}{\sqrt{2}i} \left(z^{st*}(x) + \frac{\delta}{\delta z_{st}(x)} \right)$. So realization of operators $\hat{q}_{st}(x)$ and $\hat{p}^{st}(x)$ is the holomorphic representation. Due to conditions (3) that has appeared along with Bogoliubov transformation, it is impossible to use holomorphic representation for the reduction of states numbers straightway.

And we use the following scheme:

we use Bogoliubov's transformation (2) and, in spite of appearance of exceed states, we will develop scheme of perturbation theory. Then reduction of states number will be made, so it will depend on dynamical system equations.

In the terms of new variables the operators of coordinate and momentum reads:

$$\hat{q}_{st}(x) = \frac{1}{\sqrt{2}} \left(g v_{st}(x') + u_{st}(x') + i \frac{\delta}{\delta u_n^{st}(x')} + B_q^p(a) \frac{\delta a^q}{\delta f_n^{st}(x)} \left(i S_p + i A_p^r(a) \frac{\partial}{\partial a^r} \right) \right),$$

$$\hat{p}^{st}(x) = \frac{1}{\sqrt{2}} \left(g v_n^{st}(x') + u_n^{st}(x') - i \frac{\delta}{\delta u_{st}(x')} - B_q^p(a) \frac{\delta a^q}{\delta f_{st}(x)} \left(i S_p + i A_p^r(a) \frac{\partial}{\partial a^r} \right) \right).$$

Cause $g \gg 1$, operators $\frac{\partial}{\partial \tau^\alpha}$ enter in the $\hat{q}_{st}(x')$ and $\hat{p}^{st}(x')$ in the order $O\left(\frac{1}{g}\right)$. With an eye to increase the order of the velocity, it is necessary to make a canonical transformation. Let's substitute state vector ψ on the vector:

$$\psi \longrightarrow e^{ig^2 J} \psi,$$

that accords to the substitution:

$$-i A_p^r(a) \frac{\partial}{\partial a^r} \longrightarrow g^2 J_p - i A_p^r(a) \frac{\partial}{\partial a^r}. \quad (6)$$

After canonical transformation the operators $\hat{p}^{st}(x)$ and $\hat{q}_{st}(x)$ become the following series:

$$\hat{q}_{st} = g \left(F_{st}(x') + \frac{1}{g} \hat{Q}_{st}(x') + \frac{1}{g^2} A_{st}(x') \right),$$

$$\hat{p}^{st} = g \left(F_n^{st}(x') + \frac{1}{g} \hat{P}^{st}(x') + \frac{1}{g^2} A_n^{st}(x') \right).$$

Explicit expressions for the addends in the series are:

$$\begin{aligned} F_{st}(x') &= \frac{1}{\sqrt{2}} \left(v_{st}(x') + N_{st}^k(x') J_k \right), \\ F_n^{st}(x') &= \frac{1}{\sqrt{2}} \left(v_n^{st}(x') + N_n^{stk}(x') J_k \right), \\ \hat{Q}_{st}(x') &= \frac{1}{\sqrt{2}} \left(u_{st}(x') + i \frac{\delta}{\delta u_n^{st}(x')} - N_{st}^k(x') r_k \right), \\ \hat{P}^{st}(x') &= \frac{1}{\sqrt{2}} \left(u_n^{st}(x') - i \frac{\delta}{\delta u_{st}(x')} - N_n^{stk}(x') r_k \right), \\ A_{st}(x') &= \frac{\delta a^p}{\delta f_n^{st}(x)} \left(B_p^r(a) R_r^k r_k - i K_p \right), \\ A_n^{st}(x') &= \frac{\delta a^p}{\delta f_{st}(x)} \left(B_p^r(a) R_r^k r_k - i K_p \right), \end{aligned}$$

here

$$T_c = K_c + R_c^a r_a, \quad K_p = B_p^q(a) S_q + \frac{\partial}{\partial a^p}, \quad r_k = R_k^p J_p,$$

We define contravariant components of coordinate operator and covariant component of momentum operator taking into account that they have to satisfy the following relations:

$$\hat{q}^{sl}(x) \hat{q}_{st}(x) = \delta_t^l, \quad \hat{q}^{sl}(x) \hat{p}_{st}(x) = \hat{p}^{sl}(x) \hat{p}_{st}(x).$$

It is possible if the operators reads:

$$\hat{q}^{st}(x) = g \left(F^{st}(x') - \frac{1}{g} \hat{Q}^{st}(x') + \frac{1}{g^2} B^{st}(x') \right),$$

$$\hat{p}_{st}(x) = g \left(F_{nst}(x') + \frac{1}{g} S_{st}(x') + \frac{1}{g^2} D_{st}(x') \right),$$

and the addends are defined by the following way:

$$\begin{aligned} F^{sl} F_{nst} &= \delta_t^l, \quad \hat{Q}^{st} = F^{rt} \hat{Q}_{rl} F^{sl}, \\ B^{st} &= F^{sk} \hat{Q}_{kl} F^{lt} \hat{Q}_{sm} F^{mr} - F^{st} A_{sl} F^{lr}, \quad F_n^{st} F_{sl} = F^{st} F_{nsl}, \\ S_{kl} &= F_n^{bt} \left(\hat{Q}_{bt} F_{kl} + \hat{Q}_{bl} F_{kt} \right) + F_{bl} \hat{P}^{st} F_{kt}, \\ D_{pl} &= A_{npl} + F_n^{st} (F_{tp} A_{sl} + F_{sl} A_{tp}) + F_n^{sm} \hat{Q}_{sl} \hat{Q}_{mp} + 2 F_{tp} \hat{P}^{st} \hat{Q}_{sl}. \end{aligned}$$

Then action can be represented as series with respect to inverted powers of coupling constant:

$$S = S_0 + S_1 + S_2. \tag{7}$$

5 Perturbation Theory Construction

Now we can quantize and substitute $u_{st}(x')$, $u_n^{st}(x')$ as following:

$$u_{st}(x') \mapsto \hat{q}^{st}(x), \quad u_n^{st}(x') \mapsto \hat{p}^{st}(x).$$

In the series (7) operators S_0 are C -numbers.

Let's consider the higher order.

$$S_1 = \int_{\Sigma} A_{st}(x') \hat{P}^{st}(x') + B^{st}(x') \hat{Q}_{st}(x'),$$

here explicit view of $A_{st}(x')$ and $B^{st}(x')$ are given in the Appendix 1, and operator S_1 is linear with respect to $u_{st}(x')$, $u_n^{st}(x')$, $\frac{\partial}{\partial u_{st}(x')}$, $\frac{\partial}{\partial u_n^{st}(x')}$. There are no any normalizable eigenvectors of these operators, so it is required to set them to zero for perturbation theory construction. Let's explore if it is possible.

We can write the action as :

$$S_1 = \int_{\Sigma} A_{st} \left(u_n^{st} - i \frac{\delta}{\delta u_{st}} - N_n^{stk} r_k \right) + B^{st} \left(u_{st} + i \frac{\delta}{\delta u_{st}} - N_{st}^k r_k \right).$$

We can represent actions as a sum:

$$S_1 = -i \int_{\Sigma} A_{st} \frac{\delta}{\delta u_{st}} - B^{st} \frac{\delta}{\delta u_n^{st}} + \int_{\Sigma} A_{st} \left(u_n^{st} - N_n^{stk} r_k \right) + B^{st} \left(u_{st} - N_{st}^k r_k \right).$$

Above we have got linear form with respect to $\frac{\delta}{\delta u_{st}(x')}$ and $\frac{\delta}{\delta u_n^{st}(x')}$ that is equal zero:

$$\int_{\Sigma} d\sigma \left(M_{str} \frac{\delta}{\delta u_{st}} + M_{nr}^{st} \frac{\delta}{\delta u_n^{st}} \right) = 0.$$

Linear form with respect to derivatives in S_{-1} will be equal zero if we demand $A_{st}(x')$ and $B^{st}(x')$ to be linearly connected with $M_{sta}(x')$ and $M_{na}^{st}(x')$:

$$A_{st}(x') = c^a M_{sta}(x'), \quad B^{st}(x') = c^a M_{na}^{st}(x').$$

Linear form with respect to $u_{st}(x')$ and $u_n^{st}(x')$ in the S_1 looks like

$$S_1 = \int_{\Sigma} A_{st} \left(u_n^{st} - N_n^{stk} r_k \right) + B^{st} \left(u_{st} - N_{st}^k r_k \right) = c^a \int_{\Sigma} \left(M_{sta} u_n^{st} - M_{na}^{st} u_{st} \right) - r_k \int_{\Sigma} \left(M_{sta}(x') N_n^{stk} - M_{na}^{st} N_{st}^k \right).$$

Taking into account that

$$\omega \left(M_{sta} N_{st}^k \right) = \delta_a^k$$

and

$$r_k = R_k^p J_p = J_p \int_{\Sigma} d\sigma \xi_r^i(x') \left(N_{ni}^{stp}(x') u_{st}(x') - N_{sti}^p(x') u_n^{st}(x') \right).$$

we state that the linear form with respect to $u_{st}(x')$ and $u_n^{st}(x')$ will be equal zero if the following conditions are performed on Σ : $v_{st}(x') = J_k N_{st}^k(x')$, $F_{st}(x') = \sqrt{2} v_{st}(x')$.

Here we can obtain the expression for the parameter of canonical transformation (6) (velocity of the classical part):

$$J_k = \frac{1}{\sqrt{2}} \int_{\Sigma} F_n^{st} M_{stk} - F_{st} M_{nk}^{st}.$$

We've got that $c^a = \sqrt{2}$, so we can state that linear form of derivatives with respect to $u_{st}(x')$ and $u_n^{st}(x')$ in the operator S_1 will be equal zero if the

following equations holds true on Σ :

$$F_{stn} = \frac{2a}{\sqrt{F}} \left(F_{stn} - \frac{1}{2} F_n F_{st} \right),$$

$$F_{nn}^{st} = \frac{a}{2\sqrt{F}} \left(F_{lkn} F_n^{lk} - \frac{1}{2} F_n^2 \right) F^{st} - \frac{2a}{\sqrt{F}} \left(F_n^{st} F_n^{kl} F_{stn} - \frac{1}{2} F_n F_n^{st} \right) - \quad (8)$$

$$-a\sqrt{F} \left(R^{st} - \frac{1}{2} F^{st} R \right) - \sqrt{F} \left(F^{sl} c_{;l}^t - F^{st} c_{;l}^l \right),$$

These equations could be treated as evolution equations.

Herandafter we assume $F_{st}(x)$ to be solution of the equations (8), and $F_{st}(x')$ and $F_n^{st}(x')$ on Σ are the solution of the Cauchy problem on Σ , so we can state that on the 3D manifold the evolution equation holds true. The constraint equations

$$\frac{1}{\sqrt{F}} \left(F_n^{st} F_{stn} - \frac{1}{2} F_n^2 \right) - \sqrt{F} R(F) = 0, \quad F_{;l}^{sl} = 0$$

we obtain as a conditions on the choice of Σ surface. (See (1))

So we can state that Einstein equations performance is necessary condition for the perturbation theory to be applicable. We would like to underline that Einstein equations has been obtained in the process of perturbation theory construction as a condition of validity, not as a sequence of variational principle.

6 Conclusion

We applied Bogoliubov transformation to the quantization of gravitational field in the neighbourhood of nontrivial classical component, that permitted us to avoid zero-mode problem.

Einstein equations for the classical component has been obtained as a necessary condition for the perturbation theory to be applicable.

The expression for quantum corrections of the field operator and explicit view of state si the task of the next article.

7 Appendix 1

Let's consider expansion of Hamiltonian into the series with respect to inverted powers of coupling constant:

$$H = \frac{1}{\sqrt{\gamma}} \left(\pi_{st} \pi^{st} - \frac{1}{2} \pi^2 \right) - \sqrt{\gamma} R$$

We have the order that depends only from classical part of the field:

$$H_0 = \frac{1}{\sqrt{F}} \left(F_{nst} F_n^{st} - \frac{1}{2} F_n^2 \right) - \sqrt{F} R(F),$$

the following order is linear with respect to quantum addend and derivatives with respect to quantum addends:

$$H_1 = \frac{1}{\sqrt{F}} \left(\begin{aligned} & \frac{1}{2} \left(F_{nkl} F_n^{kl} - \frac{1}{2} F_n^2 \right) F^{st} \hat{Q}_{st} + \\ & \left(\hat{P}^{st} F_{nst} + F_n^{st} S_{st} \right) - F_n \left(\hat{P}^{st} F_{nst} + \hat{Q}_{st} F_n^{st} \right) \end{aligned} \right) - \sqrt{F} \left(\frac{1}{2} F^{st} R_{st}(F) \hat{Q}_{st} - \hat{Q}^{st} R_{st}(F) + F^{st} R_{st}(F, \hat{Q}) \right),$$

and the next order depends on 2dn order of quantum addend and contains Bogoliubov variables:

$$H_2 = \frac{1}{\sqrt{F}} \left(\begin{aligned} & -\frac{1}{2} \left(F_{nkl} F_n^{kl} - \frac{1}{2} F_n^2 \right) F^{st} A_{st} + \\ & \left(A_n^{st} F_{nst} + F_n^{st} D_{st} + \hat{P}^{st} S_{st} \right) - \\ & F_n \left(A_n + A_{st} F_n^{st} + \hat{Q}_{st} \hat{P}^{st} \right) - \frac{1}{2} F_n (\hat{P} + F_n^{st} \hat{Q}_{st})^2 \end{aligned} \right) - \sqrt{F} \left(\begin{aligned} & \frac{1}{2} F^{st} \hat{Q}_{st} \left(\left(\hat{P}^{st} F_{nst} + S_{st} F_n^{st} \right) - F_n (\hat{P} + F_n^{st} \hat{Q}_{st}) \right) \\ & \frac{1}{2} R F^{st} A_{st} + R(F, \hat{Q}, A) + B^{st} R_{st} - \hat{Q}^{st} R_{st}(F, \hat{Q}) \\ & + \frac{1}{2} \left(F^{st} R(F, \hat{Q}) - \hat{Q}^{st} R_{st} \right) F^{st} \hat{Q}_{st} \end{aligned} \right).$$

Lets consider the addend $a\sqrt{F} F^{st} R_{st}(F, \hat{Q})$.

Recall that: $\Gamma_{lt}^s = \gamma^{sp} \Gamma_{ltp}$,

and Kristoffel symbols are the series:

$$\Gamma_{ltp} = \Gamma_{ltp}(F) + \frac{1}{g} \Gamma_{ltp}(\hat{Q}) + \frac{1}{g^2} \Gamma_{ltp}(A),$$

$$\text{so } \Gamma_{lt}^s = \Gamma_{lt}^s(F) + \frac{1}{g} \Gamma_{lt}^s(\hat{Q}),$$

$$\text{where } \Gamma_{lt}^s(\hat{Q}) = \left(F^{sp} \Gamma_{ltp}(\hat{Q}) - \hat{Q}^{sp} \Gamma_{ltp}(F) \right).$$

Taking into account that

$$\hat{Q}_{tp;l} = \hat{Q}_{tp;l} - \Gamma_{lt}^m(F) \hat{Q}_{mp} - \Gamma_{lp}^m(F) \hat{Q}_{mt},$$

we note:

$$\Gamma_{ltp}(\hat{Q}) = \frac{1}{2} \left(\hat{Q}_{tp;i} + \hat{Q}_{lp;i} - \hat{Q}_{lt;p} \right) - 2 \hat{Q}_{mp} \Gamma_{lt}^m(F),$$

so we can state that

$$\Gamma_{lt}^s(\hat{Q}) = \frac{1}{2} F^{sp} \left(\hat{Q}_{tp;i} + \hat{Q}_{lp;i} - \hat{Q}_{lt;p} \right),$$

and hence Ricci tensor is the series too:

$$R_{st} = R_{st}(F) + \frac{1}{g} R_{st}(F, \hat{Q}) + \frac{1}{g^2} R_{st}(F, \hat{Q}, A),$$

where

$$R_{st}(F, \hat{Q}) = \Gamma_{st;l}^l(\hat{Q}) - \Gamma_{st;t}^l(\hat{Q}) + \\ + \Gamma_{st}^l(\hat{Q})\Gamma_{lm}^m(F) + \Gamma_{st}^l(F)\Gamma_{lm}^m(\hat{Q}) - \Gamma_{sl}^m(\hat{Q})\Gamma_{mt}^l(F) - \Gamma_{sl}^m(F)\Gamma_{mt}^l(\hat{Q}),$$

so that

$$R_{st}(F, \hat{Q}) = \left(\Gamma_{st}^l(\hat{Q}) \right)_{;l} - \left(\Gamma_{st}^l(\hat{Q}) \right)_{;t}$$

and addend under consideration is

$$a\sqrt{F}F^{st}R_{st}(F, \hat{Q}) = \\ \sqrt{F} \left(aF^{st}\Gamma_{st}^l(\hat{Q}) \right)_{;l} - \sqrt{F} \left(aF^{st}\Gamma_{sl}^l(\hat{Q}) \right)_{;t} + \\ + \sqrt{F}F^{st}a_{;t}\Gamma_{sl}^l(\hat{Q}) - \sqrt{F}F^{st}a_{;l}\Gamma_{st}^l(\hat{Q}),$$

here we note

$$c^s = a_{;t}F^{st}.$$

Let's consider the form:

$$\sqrt{F}F^{st}a_{;t}\Gamma_{sl}^l(\hat{Q}) = \sqrt{F}c^s \frac{1}{2}F^{lp} \left(\hat{Q}_{sp;l} + \hat{Q}_{lp;s} - \hat{Q}_{sl;p} \right).$$

Taking into account that the expression is equal zero

$$F^{lp} \left(\hat{Q}_{sp;l} - \hat{Q}_{sl;p} \right) = 0$$

because it contains product of symmetric and antisymmetric tensors.

Analogously

$$\sqrt{F}F^{st}a_{;t}\Gamma_{sl}^l(\hat{Q}) = \frac{1}{2} \left(\sqrt{F}c^s F^{lp}\hat{Q}_{lp} \right)_{;s} - \frac{1}{2}\sqrt{F}F^{lp}\hat{Q}_{lp}c_{;s}^s$$

and

$$\sqrt{F}F^{st}a_{;l}\Gamma_{st}^l(\hat{Q}) = Div + \frac{1}{2}\sqrt{F}F^{lp}\hat{Q}_{lp}c_{;s}^s - \sqrt{F}F^{sl}\hat{Q}_{lp}c_{;s}^p,$$

and finally the addend reads:

$$a\sqrt{F}F^{st}R_{st}(F, \hat{Q}) = Div + \sqrt{F}\hat{Q}_{st} \left(F^{sp}c_{;p}^t - F^{st}c_{;p}^p \right).$$

Analogously we can state:

$$a\sqrt{F}F^{st}R_{st}(F, \hat{Q}, A) = Div + \sqrt{F}A_{st} \left(F^{sp}c_{;p}^t - F^{st}c_{;p}^p \right)$$

Lets' consider the following order of the addend:

$$a\sqrt{F}F^{st}F^{st}\hat{Q}_{st}R_{st}(F, \hat{Q}) = \\ Div + \sqrt{F}F^{st} \left(a\hat{Q} \right)_{;t} \Gamma_{sl}^l(\hat{Q}) - \sqrt{F}F^{st} \left(a\hat{Q} \right)_{;l} \Gamma_{st}^l(\hat{Q}).$$

Denote

$$r^s = \left(a\hat{Q} \right)_{;t} F^{st},$$

so straightforward calculations shows us that

$$a\sqrt{F}F^{st}\hat{Q}R_{st}(F, \hat{Q}) = \sqrt{F}\hat{Q} \left(F^{sp}r_{;p}^t - F^{st}r_{;p}^p \right) = \\ \sqrt{F} \left(F^{sp}c_{;p}^t - F^{st}c_{;p}^p \right) \hat{Q}_{st}\hat{Q}_{st}F^{st} + a\sqrt{F}F^{st} \left(F^{sp}F^{tr} - F^{st}F^{pr} \right) \hat{Q}_{st,pr},$$

and we can state that the addend is looks like the following:

$$a\sqrt{F}\hat{Q}^{st}R_{st}(F, \hat{Q}) = \sqrt{F} \left(a\hat{Q}^{st} \right)_{;t} \Gamma_{sl}^l(\hat{Q}) - \sqrt{F} \left(a\hat{Q}^{st} \right)_{;l} \Gamma_{st}^l(\hat{Q}) = \\ \sqrt{F} \left(F^{sp}a_{;tp}\hat{Q}^{st} - F^{st}a_{;tp}\hat{Q}^{pt} \right) \hat{Q}_{st} + a\sqrt{F}F^{st} \left(F^{sp}\hat{Q}_{;tp}^{st} - F^{st}\hat{Q}_{;tp}^{pt} \right) \hat{Q}_{st}.$$

So we can state that

$$\begin{aligned}
S_1 &= \hat{P}^{st}(x') F_{nst}(x') + F_n^{st}(x') \hat{Q}_{nst}(x') + a H_1(F, \hat{Q}) = \\
&= \int_{\Sigma} A_{st}(x') \hat{P}^{st}(x') + B^{st}(x') \hat{Q}_{st}(x'), \\
A_{st} &= \frac{2a}{\sqrt{F}} \left(F_{nst} - \frac{1}{2} F_n F_{st} \right), \\
B^{st} &= \frac{a}{2\sqrt{F}} \left(F_{nkl} F_n^{kl} - \frac{1}{2} F_n^2 \right) F^{st} - \frac{2a}{\sqrt{F}} \left(F_n^{st} F_n^{kl} F_{nst} - \frac{1}{2} F_n F_n^{st} \right) - \\
&- a\sqrt{F} \left(R^{st} - \frac{1}{2} F^{st} R \right) - \sqrt{F} \left(F^{sl} c_{;l}^t - F^{st} c_{;l}^l \right),
\end{aligned}$$

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